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A new class of isochronous dynamical systems

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Abstract

A new class of *isochronous* dynamical systems is introduced and briefly discussed. These systems feature in their phase space a *fully dimensional* region (part of which can be explicitly identified) where *all* their solutions are *completely periodic* (periodic in *all* their degrees of freedom) with the *same* period. But in other regions of their phase space their evolution might be quite complicated.

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1. Introduction and main results

Over the last decade quite a few *isochronous* systems—systems featuring in their phase space a *fully dimensional* region where *all* their solutions are *completely periodic* (i.e., periodic in all their degrees of freedom) with the *same* period—have been identified and investigated: for a recent review of these developments see the monograph [1]. In this paper one more class of isochronous dynamical systems is identified and briefly discussed. The relevant equations of motion read

$$\dot{z}_n = \mathrm{i}\omega Z(\underline{z})[1 + f_n(\underline{z}) - F(\underline{z})], \qquad (1a)$$

$$Z(\underline{z}) = \frac{1}{N}\sum_{n=1}^{N} z_n, \qquad (1b)$$

$$F(\underline{z}) = \frac{1}{N}\sum_{n=1}^{N} f_n(\underline{z}). \qquad (1c)$$

Notation: here and hereafter a superimposed dot denotes differentiation with respect to the (*real*) independent variable t ('time'), i is the imaginary unit ($\mathrm{i}^2 = -1$), N is an arbitrary *positive* integer (setting the number of *complex* degrees of freedom of the system), n is a positive integer which is always assumed to range from 1 to N unless otherwise indicated, ω is an arbitrary positive number to which we associate the period

$$T = \frac{2\pi}{\omega}, \tag{1d}$$

the complex coordinates $z_n \equiv z_n(t)$ are the dependent variables (also identified as the N components of the N-vector $\underline{z} \equiv \underline{z}(t)$), their time evolution taking place in the complex z-plane, and the N (generally complex) functions $f_n(\underline{z})$ depend *arbitrarily* on the N complex variables z_n, except for the (marginal) restriction that they be *analytic* (but not necessarily *holomorphic*) functions of these complex variables.

In section 3, we prove that the class of dynamical systems characterized by these equations of motion, (1), is isochronous, namely that there exists a (fully dimensional) region in the phase space of these systems—characterized by appropriate restrictions on the N complex dependent variables z_n, and of course invariant under the flow (1)—where all their solutions are completely periodic with the period T:

$$z_n(t+T) = z_n(t). \tag{2}$$

Indeed, we show that a condition on the initial data *sufficient* to guarantee this outcome is provided by the following inequality:

$$|Z_0| < \frac{W}{2(N+1)M}, \tag{3a}$$

where Z_0 (here and throughout) is the *initial* value of the center-of-mass coordinate Z, see (1),

$$Z_0 = \frac{1}{N}\sum_{n=1}^{N} z_n(0), \tag{3b}$$

the positive constant W is characterized by the requirement that all the N functions $f_n(\underline{z}_0 + \underline{w})$ be holomorphic functions of the N complex variables w_n provided $|w_n| < W$, and the positive constant M is then defined as follows:

$$M = \operatorname*{Max}_{n=1,\ldots,N;|w_n|<W} |1 + f_n(\underline{z}_0 + \underline{w}) - F(\underline{z}_0 + \underline{w})|. \tag{3c}$$

Here we used the short-hand notation $\underline{z}_0$ to denote the (complex) N-vector whose components are the initial positions $z_n(0)$. Clearly the required holomorphy of $f_n(\underline{z}_0 + \underline{w})$ (considered as functions of the components w_n of the N-vector $\underline{w}$) can always be guaranteed by assigning an adequately small value to W (provided the initial data $z_n(0)$ are assigned, as they indeed should be, where the quantities $f_n(\underline{z})$ are holomorphic in their arguments z_n); while the second requirement, (3*a*), can then also be generally satisfied, for instance, by choosing initial data such that the modulus of Z_0 is adequately small. It is thus clearly seen that there generally exists a fully dimensional region in the phase space of the dynamical system (1) (the space of complex N-vectors $\underline{z}$) such that the assignment of initial data in it yields isochronous motions (see (2)).

Let us emphasize that it is *a priori* remarkable that the *isochrony* of this class of dynamical systems obtains for such a *largely arbitrary* assignment, as described above, of the functions $f_n(\underline{z})$ characterizing this class of dynamical systems (see (1)). Outside the isochrony region the behavior of the system can be again *periodic* but with a period that is an *integer multiple* of the basic period T (see explanation and examples below) or it might be *aperiodic* indeed possibly quite complicated, requiring an investigation to be performed on a case-by-case basis.

This is implied by previous treatments; see [1] and other literature quoted there (for instance [2, 3, 5, 6]). In this respect, however, we show below an interesting negative result: in *chaotic* systems one generally finds an exponentially growing number of unstable *isolated* periodic orbits, while in our model, (1), there are *no* isolated periodic orbits. In other words, if a given initial condition $\underline{z}(0)$ lies on a periodic orbit (whether or not it is in the isochrony region), then all initial conditions that are sufficiently close to $\underline{z}(0)$ also lie on periodic orbits.

Clearly the system (1) is at equilibrium for any configuration $\underline{\bar{z}}$ such that

$$\bar{Z} = \sum_{n=1}^{N} \bar{z}_n = 0. \tag{4a}$$

Linearization of the equations of motion (1) in the neighborhood of such an equilibrium configuration, as entailed by setting

$$z_n(t) = \bar{z}_n + \varepsilon\theta_n(t) \tag{4b}$$

with ε infinitesimal, yields the equations of motion

$$\dot{\theta}_n = \mathrm{i}\omega\Theta(\underline{\theta})[1 + f_n(\underline{\bar{z}}) - F(\underline{\bar{z}})], \tag{5a}$$

$$\Theta(t) = \sum_{n=1}^{N} \theta_n(t). \tag{5b}$$

The explicit solution of the corresponding initial-value problem reads

$$\theta_n(t) = \theta_n(0) + \Theta(0)[1 + f_n(\underline{\bar{z}}) - F(\underline{\bar{z}})][\exp(\mathrm{i}\omega t) - 1], \tag{6}$$

clearly displaying its completely periodic character (with period T, see (1*d*)); consistently with the fact that initial data in the infinitesimal neighborhood of the equilibrium configuration (4*a*) are certainly within the isochrony region, as implied by the previous discussion. This solution, (6), is obtained by firstly solving the equation

$$\dot{\Theta}(t) = \mathrm{i}\omega\Theta(t) \tag{7a}$$

implied by (5) and (1*c*), the solution of which reads of course

$$\Theta(t) = \Theta(0)\exp(\mathrm{i}\omega t). \tag{7b}$$

One then replaces $\Theta(\underline{\theta})$ in (5) with this explicitly time-dependent solution and integrates the trivial system of ODEs

$$\dot{\theta}_n = \mathrm{i}\omega\Theta(0)[1 + f_n(\underline{\bar{z}}) - F(\underline{\bar{z}})]\exp(\mathrm{i}\omega t). \tag{7c}$$

Note that the system (1) allows no other equilibrium configuration besides those satisfying (4*a*), because the N equations

$$1 = F(\underline{\bar{z}}) - f_n(\underline{\bar{z}}) \tag{8}$$

characterizing such an alternative equilibrium configuration cannot be all satisfied: the sum over n from 1 to N of the left-hand sides of this set of equations yields N while, via (1*c*), the analogous sum of the right-hand sides vanishes.

In section 2, some specific examples of dynamical systems belonging to the class (1) are exhibited and tersely discussed. Proofs of the isochrony property reported in this section, and of some of the findings reported in section 2, are provided in section 3. Some other kinds of isochronous systems and some related examples are reported in section 4 (with some relevant proofs confined to the appendix).

2. Examples

First of all let us exhibit the real version of the system (1): it is obtained of course by introducing the real and imaginary parts of the complex coordinates z_n and of the functions $f_n(\underline{z})$, $z_n = x_n + \mathrm{i}y_n$, $f_n(\underline{z}) = u_n(\underline{x}, \underline{y}) + \mathrm{i}v_n(\underline{x}, \underline{y})$, and it reads

$$\dot{x}_n = -\omega[Y + X(v_n - V) + Y(u_n - U)], \tag{9a}$$

$$\dot{y}_n = \omega[X + X(u_n - U) - Y(v_n - V)], \tag{9b}$$

$$X \equiv X(\underline{x}) = \sum_{n=1}^{N} x_n, \qquad Y \equiv Y(\underline{y}) = \sum_{n=1}^{N} y_n, \tag{9c}$$

$$U \equiv U(\underline{x}, \underline{y}) = \sum_{n=1}^{N} u_n(\underline{x}, \underline{y}), \qquad V \equiv V(\underline{x}, \underline{y}) = \sum_{n=1}^{N} v_n(\underline{x}, \underline{y}). \tag{9d}$$

Let us now consider a specific class of examples out of the very large universe of such examples encompassed by the dynamical system (1).

2.1. A class of examples

This class of examples is characterized by the following assignment of the functions $f_n(\underline{z})$:

$$f_n(\underline{z}) = \sum_{k=-K_-}^{K_+} \left[g_k \sum_{m=1, m\neq n}^{N} (z_n - z_m)^{2k-1} \right]$$
$$= \sum_{k=1}^{K_+} \left[\sum_{m=1, m\neq n}^{N} \left(\frac{z_n - z_m}{a_k} \right)^{2k-1} \right] + \sum_{k=0}^{K_-} \left[\sum_{m=1, m\neq n}^{N} \left(\frac{b_k}{z_n - z_m} \right)^{2k+1} \right], \tag{10}$$

where K_+ and K_- are two arbitrary nonnegative integers and the $K_+ + K_- + 1$ (generally complex) 'coupling constants' g_k are as well arbitrary. The motivation for the second version of this formula—entailing an obvious relationship among the coupling constants g_k , a_k and b_k—will soon be clear. Note that this assignment of the functions $f_n(\underline{z})$ entails $F(\underline{z}) = 0$ (see (1*c*)).

To write in a (possibly) more interesting form the real version of the isochronous dynamical system (1) with this assignment of the functions $f_n(\underline{z})$, we now introduce the following *real* two-vectors:

$$\vec{r}_n \equiv (x_n, y_n) \equiv (\mathrm{Re}\, z_n, \mathrm{Im}\, z_n), \qquad \vec{a}_k \equiv (\mathrm{Re}\, a_k, \mathrm{Im}\, a_k), \qquad \vec{b}_k \equiv (\mathrm{Re}\, b_k, \mathrm{Im}\, b_k). \tag{11}$$

The first of these formulae allows us to reinterpret the evolutions of the complex variables $z_n(t)$ in the complex z-plane as motions of *real* points described by two-vectors $\vec{r}_n(t)$ and moving in the *real* horizontal plane spanned by the real two-vector $\vec{r} \equiv (x, y)$. It is then a matter of trivial if perhaps tedious algebra to verify that the *real* version of the equations of motion (1) with (10) can thereby be re-formulated as the following set of *covariant* (and *translation-invariant*) ODEs for two-vectors lying in the horizontal plane:

$$\dot{\vec{r}}_n = \omega\{-\vec{R}\Phi_n(\underline{\vec{r}}) + \hat{k} \wedge \vec{R}[1 + \Psi_n(\underline{\vec{r}})]\}, \qquad \vec{R} \equiv \frac{1}{N}\sum_{n=1}^{N} \vec{r}_n, \tag{12a}$$

where $\hat{k} \equiv (0, 0, 1)$ denotes the unit vector orthogonal to the horizontal plane so that $\hat{k} \wedge \vec{r} \equiv (-y, x)$ (via the usual interpretation of the symbol $\wedge$ as denoting the *three-dimensional* vector product), and

$$\Psi_n(\vec{r}) = u_n(\vec{r}) = \sum_{m=1,m\neq n}^{N} \left[\sum_{k=1}^{K_+} \left(\frac{r_{nm}}{a_k} \right)^{2k-1} T_{2k-1} \left(\frac{\vec{a}_k \cdot \vec{r}_{nm}}{a_k r_{nm}} \right) + \sum_{k=0}^{K_-} \left(\frac{r_{nm}}{b_k} \right)^{-(2k+1)} T_{2k+1} \left(\frac{\vec{b}_k \cdot \vec{r}_{nm}}{b_k r_{nm}} \right) \right], \tag{12b}$$

$$\Phi_n(\vec{r}) = v_n(\vec{r}) = \sum_{m=1,m\neq n}^{N} \left[\sum_{k=1}^{K_+} \frac{r_{nm}^{2k-2}}{a^{2k}} U_{2k-2} \left(\frac{\vec{a}_k \cdot \vec{r}_{nm}}{a_k r_{nm}} \right) \hat{k} \cdot \vec{a}_k \wedge \vec{r}_{nm} - \sum_{k=0}^{K_-} \frac{r_{nm}^{-(2k+2)}}{b_k^{-2k}} U_{2k} \left(\frac{\vec{b}_k \cdot \vec{r}_{nm}}{b_k r_{nm}} \right) \hat{k} \cdot \vec{b}_k \wedge \vec{r}_{nm} \right]. \tag{12c}$$

Here (and below) we use the following obvious notation: the symbols $T_k(x)$ and $U_k(x)$ refer to the Chebyshev polynomials of order k of the first and second kinds, respectively (for notation see [9]), a dot sandwiched between two two-vectors denotes the standard (*rotation-invariant*) scalar product in the plane ($\vec{r}^{(1)} \cdot \vec{r}^{(2)} \equiv x^{(1)}x^{(2)} + y^{(1)}y^{(2)}$), likewise the notation $\hat{k} \cdot r^{(1)} \wedge \vec{r}^{(2)}$ denotes the standard (rotation-invariant) pseudoscalar product in the plane ($\hat{k} \cdot \vec{r}^{(1)} \wedge \vec{r}^{(2)} \equiv x^{(1)}y^{(2)} - x^{(2)}y^{(1)}$), and of course $a_k^2 \equiv \vec{a}_k \cdot \vec{a}_k$, $r_{nm}^2 \equiv \vec{r}_{nm} \cdot \vec{r}_{nm}$, $\vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m$.

2.2. *A rather trivial solvable case*

A very simple special case belonging to the class (10) obtains for

$$f_n(\underline{z}) = \sum_{m=1,m\neq n}^{N} \left(\frac{z_n - z_m}{a} \right), \tag{13a}$$

yielding the (*real*, *covariant*) equations of motion

$$\dot{\vec{r}}_n = \omega\{-\vec{R}\Phi_n(\underline{\vec{r}}) + \hat{k} \wedge \vec{R}[1 + \Psi_n(\underline{\vec{r}})]\}, \qquad \vec{R} \equiv \frac{1}{N} \sum_{n=1}^{N} \vec{r}_n, \tag{13b}$$

$$\Phi_n(\underline{\vec{r}}) = \sum_{m=1,m\neq n}^{N} \frac{\hat{k} \cdot \vec{a} \wedge \vec{r}_{nm}}{a^2}, \qquad \Psi_n(\underline{\vec{r}}) = \sum_{m=1,m\neq n}^{N} \frac{\vec{a} \cdot \vec{r}_{nm}}{a^2}. \tag{13c}$$

In this special case, the initial-value problem can be explicitly solved (as explained in section 3). For simplicity we write the solution in terms of the complex variables and coupling constants:

$$z_n(t) = z_n(0)\exp(\sigma) + Z_0[1 - \exp(\sigma) + \sigma],$$
$$\sigma \equiv \sigma(t) = N\frac{Z_0}{a}[\exp(\mathrm{i}\omega t) - 1], \qquad Z_0 = \sum_{n=1}^{N} z_n(0). \tag{14}$$

In this case the isochrony region coincides clearly with the entire phase space.

2.3. Another trivially solvable case: examples of nonisochronous behavior

Now we look at another trivial case, which is, however, general enough to show the type of nontrivial behavior that may arise outside the isochrony region. Consider the dynamical system characterized by the equations of motion

$$\dot{z}_1 = \mathrm{i}\omega Z[1 + f(z_1 - z_2)], \qquad \dot{z}_2 = \mathrm{i}\omega Z[1 - f(z_1 - z_2)], \tag{15}$$

where $f(z)$ is an arbitrary function. This is clearly an instance of (1) with $N = 2$, in which one takes $f_n(\underline{z}) = (-1)^{n+1} f(z_1 - z_2)$ depending only on $z_1 - z_2$ but otherwise arbitrary. Then clearly (via (1*b*), or see section 3 below)

$$Z(t) = Z(0)\exp(\mathrm{i}\omega t) \equiv Z_0 \exp(\mathrm{i}\omega t), \tag{16}$$

and so if one introduces

$$z(t) = z_1(t) - z_2(t), \tag{17a}$$

entailing of course

$$z_1(t) = Z(t) + \frac{z(t)}{2}, \qquad z_2(t) = Z(t) - \frac{z(t)}{2}, \tag{17b}$$

one obtains

$$\dot{z} = 2\mathrm{i}\omega Z_0 \exp(\mathrm{i}\omega t) f(z). \tag{18}$$

Since $Z(t)$ is clearly periodic with period T, deviations from isochrony can only be found in the time evolution of $z(t)$.

Two simple examples show possible behaviors different from isochrony. Let us consider first the case $f(z) = \left(\frac{z}{a}\right)^{\lambda}$, with λ an arbitrary real number different from unity and a an arbitrary constant. Then clearly

$$z(t) = \{[z(0)]^{1-\lambda} + 2(1-\lambda)a^{-\lambda} Z_0[\exp(\mathrm{i}\omega t) - 1]\}^{1/(1-\lambda)}. \tag{19}$$

It is thus seen that, as long as there holds the inequality (characterizing the isochrony region of the initial data)

$$|Z_0| < \left| Z_0 - \frac{a^{\lambda}[z(0)]^{1-\lambda}}{2(1-\lambda)} \right|, \tag{20}$$

solution (19) is periodic with period T (see (1*d*)). On the other hand when this inequality, (20), is reversed, solution (19) is *periodic* with a period which is an integer multiple of T if λ is *rational*, and it is instead *aperiodic* (in fact, *quasiperiodic*) if λ is *irrational*. It is thus seen that aperiodic behaviors are indeed possible.

Next, let us consider the case $f(z) = \exp(z/a)$, with a an arbitrary (of course *nonvanishing*) constant. Then the solution reads

$$z(t) = z(0)\ln\left\{1 - \frac{2Z_0}{a}\exp\left[\frac{z(0)}{a}\right][\exp(\mathrm{i}\omega t) - 1]\right\}, \tag{21}$$

hence it is periodic with period T provided there holds the inequality

$$|Z_0| < \left| Z_0 + \frac{a}{2}\exp\left[-\frac{z(0)}{a}\right]\right|, \tag{22}$$

but it is unbounded (diverging linearly in t) when this inequality is reversed. It is thus seen that unbounded behaviors are also possible. Note that these two regions of initial data, each having full dimensionality in phase space, are separated by a *lower-dimensional* region of initial data (characterized by the equation that obtains by replacing the inequality sign in (22) with an equality sign) yielding solutions that become *singular* at real values of t.

The question of the degree of complexity that may emerge is a more difficult one. However, a common occurrence among systems displaying complex behavior, specifically among *chaotic* systems, is the presence of a large number of *isolated* unstable periodic orbits. As we show below, such isolated periodic orbits can generally be excluded for all systems of type (1). Thus, this specific type of chaotic behavior is excluded.

2.4. A less trivial, yet also solvable, case

A more general yet also *solvable* case, also belonging to the class (10), is

$$f_n(\underline{z}) = \sum_{m=1, m\neq n}^{N} \left(\frac{z_n - z_m}{a} + \frac{b}{z_n - z_m} \right), \tag{23a}$$

yielding the (real, covariant) equations of motion

$$\dot{\vec{r}}_n = \omega\{-\vec{R}[\Phi_n(\underline{\vec{r}}) + \tilde{\Phi}_n(\underline{\vec{r}})] + \hat{k} \wedge \vec{R}[1 + \Psi_n(\underline{\vec{r}}) + \tilde{\Psi}_n(\underline{\vec{r}})]\}, \qquad \vec{R} \equiv \frac{1}{N}\sum_{n=1}^{N} \vec{r}_n, \tag{23b}$$

with the quantities $\Phi_n(\underline{\vec{r}})$ and $\Psi_n(\underline{\vec{r}})$ defined as above (see (13c)), and

$$\tilde{\Phi}_n(\underline{\vec{r}}) = -\sum_{m=1, m\neq n}^{N} \frac{\hat{k} \cdot \vec{b} \wedge \vec{r}_{nm}}{r_{nm}^2}, \qquad \tilde{\Psi}_n(\underline{\vec{r}}) = \sum_{m=1, m\neq n}^{N} \frac{\vec{b} \cdot \vec{r}_{nm}}{r_{nm}^2}. \tag{23c}$$

In this special case the initial-value problem can be solved, not quite explicitly but in the usual sense of reducing it to the following sequence of purely algebraic operations (as explained in section 3). Again, for simplicity, we provide here the solution of the initial-value problem in terms of the complex variables and coupling constants: the N coordinates $z_n(t)$ are given by the formula

$$z_n(t) = Z_0 \exp(\mathrm{i}\omega t) + \xi_n(t), \tag{24a}$$

where the N (complex, time-dependent) numbers $\xi_n(t)$ are the N zeros of the (time-dependent, monic) polynomial, of degree N in the variable ξ,

$$P_N(\xi, t) = \left(\frac{4}{\gamma}\right)^{N/2} H_N(\gamma\xi) + \sum_{m=1}^{N} [\beta_m(t) H_{N-m}(\gamma\xi)] = \prod_{n=1}^{N} [\xi - \xi_n(t)], \tag{24b}$$

where $H_n(x)$ is the standard Hermite polynomial of degree n in the variable x,

$$\gamma = -\frac{N Z_0}{ab}, \tag{24c}$$

and

$$\beta_m(t) = \beta_m(0) \exp\left\{ m N \frac{Z_0}{a} [\exp(\mathrm{i}\omega t) - 1] \right\}. \tag{24d}$$

The N coefficients $\beta_m(0)$ are of course determined by the initial data, namely by the polynomial identity

$$\left(\frac{4}{\gamma}\right)^{N/2} H_N(\gamma\xi) + \sum_{m=1}^{N} [\beta_m(0) H_{N-m}(\gamma\xi)] = \prod_{n=1}^{N} [\xi - \xi_n(0)] = \prod_{n=1}^{N} \{\xi - [z_n(0) - Z_0]\}. \tag{24e}$$

It is plain (see (24d)) that the polynomial $P_N(\xi, t)$ (see (24b)) is periodic in t with period T, see (1d); hence the set of its zeros is also periodic with this same period. However, due to the possibility that through the time-evolution different zeros exchange their roles, from this explicit solution one can only conclude that the time evolution of each of the complex coordinates $z_n(t)$ is periodic with a period $T_n = k_n T$, where k_n is a positive integer not larger than N; of course there shall be sets of initial data (characterized by a sufficiently small value of $|Z_0|$) such that each coordinate $z_n(t)$ is periodic with period T, see (2).

3. Proofs

In this section, we report the proofs of those results that have been reported but not proven above.

The first task is to prove that the class of dynamical systems (1) is isochronous. The first step to prove this result is to sum over n from 1 to N the system of ODEs (1*a*). Via (1*b*) and (1*c*) one gets thereby

$$\dot{Z} = \mathrm{i}\omega Z(\underline{z}), \tag{25a}$$

yielding

$$Z(t) = Z_0 \exp(\mathrm{i}\omega t), \tag{25b}$$

where Z_0 is clearly the initial value of the center-of-mass coordinate Z (see (3*b*)). It is now convenient to introduce the (dimensionless, complex) variable

$$\tau \equiv \tau(t) = \exp(\mathrm{i}\omega t) - 1, \tag{26a}$$

entailing of course

$$\tau(0) = 0. \tag{26b}$$

This quantity is clearly periodic in the time t with period T, see (1*d*),

$$\tau(t + T) = \tau(t); \tag{26c}$$

indeed as the real variable t varies over one period (say, from 0 to T), the complex variable $\tau(t)$ makes one full round in the complex τ-plane over the circle C, of unit radius and centered at $\tau = -1$. And since clearly $\mathrm{i}\omega Z(t) = Z_0\dot{\tau}(t)$, this suggests setting

$$z_n(t) = \zeta_n(\tau), \tag{27a}$$

so that the system (1*a*) can be reformulated as a dynamical system with τ as independent variable as follows:

$$\zeta_n' = Z_0[1 + f_n(\underline{\zeta}) - F(\underline{\zeta})], \tag{27b}$$

where of course the appended prime denotes differentiation with respect to the new independent variable τ.

It is now clear that to every solution $\underline{\zeta}(\tau)$ of this (autonomous) system of ODEs that is holomorphic in τ (i.e., such that all its N components $\zeta_n(\tau)$ are holomorphic functions of the complex variable τ) in the (closed) disc encircled by the circle C in the complex τ-plane, there corresponds via (27*a*) a solution $\underline{z}(t)$ of our original system (1) that is completely periodic with period T; see (2) with (1*d*). Actually the requirement that $\zeta_n(\tau)$ be holomorphic is sufficient but not necessary: a *meromorphic* function $\zeta_n(\tau)$ having poles inside the circle C (but not on it) would still yield via (27*a*) a periodic function $z_n(t)$, see (2), as would a solution $\zeta_n(\tau)$ having essential singularities inside that circle C; moreover a function $\zeta_n(\tau)$ featuring only a *finite* number of *rational* branch points inside the circle C—on the sheets of the Riemann surface associated with that function $\zeta_n(\tau)$ due to those branch points—would clearly still yield via (27*a*) a function $z_n(t)$ depending *periodically* on the (real) independent variable t

(time), although the period would then be an integer multiple of the basic period T; and this would be the case even if the Riemann surface associated with the function $\zeta_n(\tau)$ possessed an *infinite* number of sheets generated by an infinite number of *rational* branch points lying inside the circle C (located on the infinite sheets of this Riemann surface), but only a *finite* number of these sheets would be accessed by traveling round and round the circle C, so that after a *finite* number of complete turns one would be led back to the starting point of this journey over the Riemann surface.

On the other hand, the solutions $\underline{\zeta}(\tau)$ of the (autonomous) system (27*b*)—as characterized by initial data assigned at $t = \tau = 0$—are certainly (see, for instance, section 12.21 of [7]) holomorphic in a circular disc D centered at $\tau = 0$ in the complex τ-plane, whose radius ρ_D is bounded as follows:

$$\rho_D \geqslant \rho = \frac{W}{(N+1)|Z_0|M}, \tag{28}$$

where the positive constant W is characterized by the requirement that all the N functions $f_n(\underline{z}(0) + \underline{w})$ be holomorphic functions of the complex variables w_n provided $|w_n| < W$, and the positive constant M is defined by (3*c*). Note that these quantities W and M depend on the initial data $\underline{z}(0)$, and of course as well on the functions $f_n(\underline{z})$ characterizing the dynamical system (1). Likewise for ρ. Observe now that the circle C, as defined above, is clearly enclosed inside this disc D, provided $\rho > 2$. Hence the condition (3*a*) on the initial data $\underline{\zeta}(0) = \underline{z}(0)$ implying this inequality, $\rho > 2$, guarantees that the corresponding solutions of the original system (1) are isochronous with period T. And clearly such a condition is generally satisfied by a set of initial data filling a fully dimensional region of phase space, and in particular it is satisfied if there holds in inequality (3*a*).

Our first task is thus completed.

Next we tersely indicate how the explicit solution (14) of section 2.2 is obtained. The trick is of course to follow the same procedure as in the proof given just above, hence to focus on the solution of the system of ODEs (27*b*) as an intermediate step to arrive at the solution of the original system (1). In the case treated in section 2.2 the equations of motion (27*b*) become merely an autonomous set of *linear* ODEs, whose solution is easily performed by standard techniques.

Likewise, the first step to solve the dynamical system of section 2.3 is to transform the corresponding equations of motion (i.e., (1) with (23*a*)), via (27*a*) and (1*b*), into

$$\zeta_n' = Z_0\left[1 + N\frac{\zeta_n - Z}{a} + \sum_{m=1,m\neq n}^{N}\frac{b}{\zeta_n - \zeta_m}\right], \tag{29a}$$

hence, via (25*b*) and (26*a*), into

$$\zeta_n' = Z_0\left[1 - N\frac{Z_0}{a}(1+\tau) + N\frac{\zeta_n}{a} + \sum_{m=1,m\neq n}^{N}\frac{b}{\zeta_n - \zeta_m}\right], \tag{29b}$$

hence, by setting

$$\zeta_n(\tau) = Z_0(1+\tau) + \xi_n(\tau), \tag{29c}$$

into

$$\xi_n' = Z_0\left[N\frac{\xi_n}{a} + \sum_{m=1,m\neq n}^{N}\frac{b}{\xi_n - \xi_m}\right]. \tag{29d}$$

One then recognizes that this system of ODEs belongs to the class of *solvable* dynamical systems treated in section 2.3.4.1 of [8], indeed up to trivial notational changes it coincides with equation (2.3.4.1-1) of that book. The derivation of the final result, as reported above (in section 2), is an immediate consequence.

Finally, we prove the statement made in section 1 and repeated in section 2.3, namely that there can be no isolated periodic orbits in systems of the type defined by (1). First, we show that any periodic orbit must have a period which is a rational multiple of T, except for the equilibrium solutions for which $Z(t)$ vanishes identically, which were described in section 1. Let us first assume the opposite, that is, that $\tilde{\underline{z}}(t)$ is an orbit with primitive period αT, where α is an irrational real number. Then $Z(t)$ has both period T and αT hence is, by a standard theorem, constant. If $Z(t)$ vanishes identically, we are indeed dealing with the equilibrium solutions described above. If instead $Z(t) = Z_0$ is a *nonvanishing constant*, one finds for $z_n(t)$ the equation

$$\dot{z}_n = \mathrm{i}\omega Z_0[1 + f_n(\underline{z}) - F(\underline{z})], \tag{29e}$$

which is contradictory, since it implies (by summing over n from 1 to N, see (1*b*) and (1*c*)) that $\dot{Z} = \mathrm{i}\omega Z_0$ entailing that $Z(t)$ is *not constant*.

Any periodic orbit $\tilde{\underline{z}}(t)$ has therefore a (possibly non-primitive) period pT, where p is a positive integer, since any orbit with period pT/q (with q a positive integer) also has period pT. Therefore $\tilde{\underline{z}}(t)$ is a one-valued function of $\tau^{1/p}$ (see (26*a*)). Hence all orbits in the vicinity of $\tilde{\underline{z}}(t)$ are also one valued as a function of $\tau^{1/p}$ and hence periodic with period pT.

4. Other isochronous systems

Another class of isochronous systems is characterized by the equations of motion

$$\dot{z}_n = \mathrm{i}\omega z_n[1 + f_n(\underline{z}) - F(\underline{z})], \tag{30}$$

with $F(\underline{z})$ defined again by (1*c*). It is easily seen that for this class the quantity

$$P(t) = \prod_{n=1}^{N} z_n(t) \tag{31a}$$

evolves as follows:

$$\dot{P} = \mathrm{i}N\omega P, \tag{31b}$$

entailing

$$P(t) = P(0)\exp(\mathrm{i}N\omega t) \equiv P_0 \exp(\mathrm{i}N\omega t). \tag{31c}$$

It is also easy to show that this system, (30), generally yields isochronous motions if the functions $f_n(\underline{z})$ (hence as well $F(\underline{z})$) scale *rationally*, i.e.

$$f_n(\alpha\underline{z}) = \alpha^{\lambda} f_n(\underline{z}), \tag{32}$$

with $\lambda = p/q$ be a rational number (the same for all n values). (Actually this result does not require the subtraction of the function $F(\underline{z})$; see the right-hand side of (30)), which is instead essential for the validity of (31*b*), hence (31*c*)). We only outline below (in section 4.1) the relevant proof of isochrony, since it is rather closely connected with previous findings [1]. A simple example with $N = 2$ for which the solution of the initial-value problem can be exhibited explicitly is provided in section 4.2. Note that the condition (32) is sufficient (as proven below) but *not necessary* for isochrony; indeed we exhibit below another example with $N = 2$, characterized by analogous equations of motion (see (30)) but with functions that do *not* satisfy the scaling property (32), whose initial-value problem can be explicitly solved, thereby showing that it also features an isochronous evolution.

4.1. Proof of isochrony

In this section, we outline the proof that the dynamical system (30)—even in its generalized version with $F(\underline{z})$ set to zero, but with (32)—is isochronous. To this end it is convenient to perform the following change of dependent and independent variables:

$$z_n(t) = \exp(\mathrm{i}\omega t)\varphi_n(\theta), \qquad \theta = \exp(\mathrm{i}\beta\omega t), \qquad \beta = \lambda + 1 = \frac{p+q}{q}. \tag{33a}$$

It is then easily seen (using (32)) that the equations of motion for the new dependent variables $\varphi_n(\theta)$ read

$$\varphi_n'(\theta) = \frac{\mathrm{i}\omega q}{p+q}\varphi_n(\theta) f_n[\underline{\varphi}(\theta)], \tag{33b}$$

where of course the appended prime denotes differentiation with respect to the new independent variable θ. But this equation is *autonomous*, hence there generally is an open, fully dimensional set of initial data $z_n(0) = \varphi_n(1)$ such that all the solutions $\varphi_n(\theta)$ are holomorphic functions of the complex variable θ in a circle of radius unity centered at $\theta = 1$ in the complex θ-plane. And this clearly implies that all the corresponding functions $z_n(t)$, see (33a), are periodic in t with a period which is the minimum common multiple among $T = 2\pi/\omega$ and $\tilde{T} = 2\pi/(\beta\omega) = [q/(p+q)]T$.

4.2. Two solvable examples

Here we exhibit two examples belonging to the class (30) with $N = 2$. In both cases the time evolution can be explicitly solved (this may be simply verified by direct substitution of the solutions exhibited below, or see the appendix to understand how these solutions were arrived at).

The first example is characterized by the assignment $f_1(\underline{z}) = (z_1/c_1)^\lambda + (z_2/c_2)^\lambda = -f_2(\underline{z})$ entailing $F(\underline{z}) = 0$, which of course satisfies the scaling condition (32). Here and hereafter c_1 and c_2 are two arbitrary (nonvanishing) constants, possibly complex, and λ is a real number. The equations of motion read

$$\dot{z}_1 = \mathrm{i}\omega z_1[1 + (z_1/c_1)^\lambda + (z_2/c_2)^\lambda], \qquad \dot{z}_2 = \mathrm{i}\omega z_2[1 - (z_1/c_1)^\lambda - (z_2/c_2)^\lambda], \tag{34}$$

and the solution of the initial-value problem is provided by the following formulae:

$$z_1(t) = z_1(0)\left[\frac{1+\eta\tan\xi}{1+\eta^{-1}\tan\xi}\right]^{1/\lambda}\exp(\mathrm{i}\omega t) \tag{35a}$$

$$z_2(t) = z_2(0)\left[\frac{1+\eta^{-1}\tan\xi}{1+\eta\tan\xi}\right]^{1/\lambda}\exp(\mathrm{i}\omega t), \tag{35b}$$

$$\eta = \left[\frac{c_1 z_2(0)}{c_2 z_1(0)}\right]^{\lambda/2}, \tag{35c}$$

$$\xi = \left[\frac{z_1(0)z_2(0)}{c_1 c_2}\right]^{\lambda/2}[1 - \exp(\mathrm{i}\lambda\omega t)]. \tag{35d}$$

Clearly all these solutions are periodic with period T for arbitrary λ, provided the initial data are such that all the points $\xi_\pm$ satisfying the equations

$$\tan\xi_\pm = \eta^{\pm 1} \tag{36}$$

fall, in the complex ξ-plane, outside the circle C whose diameter has one end at $\xi = 0$ and the other at $2\left[\frac{z_1(0)z_2(0)}{c_1c_2}\right]^{\lambda/2}$. And it is moreover clear that all solutions are periodic if λ is a rational (real) number, except those yielded by the lower-dimensional set of initial data such that one of the two equations (36) is satisfied for a value of ξ located exactly on the circle C. Let us also note that this complex dynamical system, (34), can also be rewritten as a real covariant system describing the motion of two points in the horizontal plane, as identified by the two-vectors $\vec{r}_n \equiv (x_n, y_n)$ with $z_n = x_n + \mathrm{i}y_n$. Restricting for simplicity consideration to the case with $\lambda = 1$ we exhibit below the corresponding equations of motion:

$$\dot{\vec{r}}_n = \omega\hat{k} \wedge \vec{r}_n \left\{1 - (-1)^n \left[\frac{\vec{\rho}_1 \cdot \vec{r}_1 + \hat{k} \cdot (\vec{\rho}_1 \wedge \vec{r}_1)}{\rho_1^2} + \frac{\vec{\rho}_2 \cdot \vec{r}_1 + \hat{k} \cdot (\vec{\rho}_2 \wedge \vec{r}_1)}{\rho_2^2}\right]\right\}, \qquad n = 1, 2, \tag{37a}$$

with the two real constant two-vectors $\vec{\rho}_n$ related to the two complex constants c_n as follows:

$$\vec{\rho}_n = (a_n, b_n), \qquad c_n = a_n + \mathrm{i}b_n, \tag{37b}$$

entailing

$$\rho_n^2 = |c|^2 = a^2 + b^2, \tag{37c}$$

and with $\hat{k} \equiv (0, 0, 1)$ the unit vector orthogonal to the horizontal plane so that $\hat{k} \wedge \vec{r}_n = (-y_n, x_n)$. And let us also display (in view of possible applications) these equations of motion componentwise in full detail:

$$\dot{x}_1 = \omega\left[-y_1 - \frac{a(2x_1y_1 - x_1y_2 - x_2y_1) + b\left(x_1^2 - y_1^2 - x_1x_2 + y_1y_2\right)}{a^2 + b^2}\right], \tag{38a}$$

$$\dot{y}_1 = \omega\left[x_1 - \frac{a\left(x_1^2 - y_1^2 - x_1x_2 + y_1y_2\right) - b(2x_1y_1 - x_1y_2 - x_2y_1)}{a^2 + b^2}\right], \tag{38b}$$

$$\dot{x}_2 = \omega\left[-y_2 - \frac{a(2x_2y_2 - x_1y_2 - x_2y_1) + b\left(x_2^2 - y_2^2 - x_1x_2 + y_1y_2\right)}{a^2 + b^2}\right], \tag{38c}$$

$$\dot{y}_2 = \omega\left[x_2 - \frac{a\left(x_2^2 - y_2^2 - x_1x_2 + y_1y_2\right) - b(2x_2y_2 - x_1y_2 - x_2y_1)}{a^2 + b^2}\right]. \tag{38d}$$

The second example is characterized by the assignment $f_1(\underline{z}) = (z_1/c_1)^\lambda + (z_2/c_2)^{-\lambda} = -f_2(\underline{z})$, again entailing $F(\underline{z}) = 0$. The equations of motion now read

$$\dot{z}_1 = \mathrm{i}\omega z_1[1 + (z_1/c_1)^\lambda + (z_2/c_2)^{-\lambda}], \qquad \dot{z}_2 = \mathrm{i}\omega z_2[1 - (z_1/c_1)^\lambda - (z_2/c_2)^{-\lambda}], \tag{39}$$

and the solution of the initial-value problem is provided by the following formulae:

$$z_1(t) = z_1(0)\exp(\mathrm{i}\omega t)\left\{1 - \left[\frac{z_1(0)}{c_1}\right]^\lambda [\exp(\mathrm{i}\lambda\omega t) - 1] + \left[\frac{z_2(0)}{c_2}\right]^{-\lambda} [\exp(-\mathrm{i}\lambda\omega t) - 1]\right\}^{-1/\lambda}, \tag{40a}$$

$$z_2(t) = z_2(0)\exp(\mathrm{i}\omega t)\left\{1 - \left[\frac{z_1(0)}{c_1}\right]^\lambda [\exp(\mathrm{i}\lambda\omega t) - 1] + \left[\frac{z_2(0)}{c_2}\right]^{-\lambda} [\exp(-\mathrm{i}\lambda\omega t) - 1]\right\}^{1/\lambda}. \tag{40b}$$

Note that in this case, provided λ is real and rational, $\lambda = p/q$, (almost) all solutions are periodic with (possibly nonprimitive) period pT, see (1*d*), with the sole exception of those corresponding to the special initial data for which the right-hand side of the above expression of $z_1(t)$ diverges at a real value of the time t.

Appendix A.

In this appendix we show that the formulae (35) respectively (40) provide the solutions of the initial-value problems for the dynamical systems (34) respectively (39).

The first observation is that (34) entail

$$\frac{\mathrm{d}}{\mathrm{d}t}[z_1(t)z_2(t)] = 2\mathrm{i}\omega[z_1(t)z_2(t)], \tag{A.1a}$$

hence

$$z_1(t)z_2(t) = z_1(0)z_2(0)\exp(2\mathrm{i}\omega t). \tag{A.1b}$$

It is then convenient to set

$$\frac{z_1(t)}{z_2(t)} = \frac{c_1}{c_2}[s(t)]^{2/\lambda}, \tag{A.2}$$

entailing via (A.1*b*)

$$z_1(t) = \left(\frac{c_1}{c_2}\right)^{1/2}[z_1(0)z_2(0)]^{1/2}[s(t)]^{1/\lambda}\exp(\mathrm{i}\omega t), \tag{A.3a}$$

$$z_2(t) = \left(\frac{c_2}{c_1}\right)^{1/2}[z_1(0)z_2(0)]^{1/2}[s(t)]^{-1/\lambda}\exp(\mathrm{i}\omega t), \tag{A.3b}$$

and via (34) and (A.3*b*)

$$\frac{\dot{s}}{1+s^2} = C\mathrm{i}\lambda\omega\exp(\mathrm{i}\lambda\omega t). \tag{A.4}$$

The integration of this last ODE is trivial and (via (A.3*a*) and (A.3*b*)) it yields solution (35).

Likewise, to integrate the equations of motion (39) we proceed just as we did above (in this appendix), arriving thereby, rather than to (A.4), to the ODE

$$\frac{\dot{s}}{s^2} = \mathrm{i}\lambda\omega[C\exp(\mathrm{i}\lambda\omega t) + C^{-1}\exp(-\mathrm{i}\lambda\omega t)], \tag{A.5}$$

and the remaining steps to arrive at solutions (40) are then elementary.

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